Calculus 1
Final Exam – Solutions
November 3, 2022 (8:30 – 10:30)

1) Apply l'Hospital's Rule to find the limit $\lim_{x\to 0^+} \sin x \ln x$. Indicate the results (e.g. continuity, differentiation rules, limit laws) used in each step.

Solution. The limit is an indeterminate form of type "$0 \cdot (-\infty)$" since $\sin 0 = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$. Thus we cannot directly apply l'Hospital's Rule. Notice however that moving $\sin x$ to the denominator by writing

$$\sin x \ln x = \frac{\ln x}{\frac{1}{\sin x}},$$

the expression is now an indeterminate form of "$0/0$". We note that we are allowed to do this because when x is around (but not equal to) 0, we have $\sin x \neq 0$ and $1/\sin x$ exists. Let us now use l'Hospital's Rule:

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} \overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} = \lim_{x\to 0^+} \left(-\frac{\sin x}{x} \frac{\sin x}{\cos x} \right) = -\lim_{x\to 0^+} \frac{\sin x}{x} \lim_{x\to 0^+} \frac{\sin x}{\cos x} = -1 \cdot \frac{\sin 0}{\cos 0} = 0.$$

Above we used the Chain Rule, the derivatives $(\ln x)' = 1/x$, $(\sin x)' = \cos x$, and the Generalized Power Rule $(x^{-1})' = -x^{-2}$. For the equality in the middle, we used the Constant Multiple and Product Laws of Limits as well as the special trigonometric limit $\frac{\sin x}{x} \to 1$ as $x \to 0$. The second to last equality follows from the continuity of the sine and cosine functions and the Quotient Law of Limits. Finally, we used that $\sin 0 = 0$ and $\cos 0 = 1 \neq 0$. Therefore we have found that

$$\lim_{x\to 0^+} \sin x \ln x = 0.$$

2) Consider the function $V(x) = ax^{-2} - bx^{-1}$, where $x > 0$ and a, b are positive constants. Compute the 2nd-degree Taylor polynomial for V centred around the point where it attains its minimum.

Solution. To locate the minimum of V we need to determine its critical numbers. The derivative of $V(x)$ vanishes at $x = 2a/b$:

$$V'(x) = -2ax^{-3} + bx^{-2} = 0 \quad \underset{\div x^3}{\overset{\times x^3}{\rightleftarrows}} \quad -2a + bx = 0 \quad \underset{\times b, -2a}{\overset{+2a, \div b}{\rightleftarrows}} \quad x = \frac{2a}{b}.$$

This is the only critical number for $x > 0$ (the domain of V). Here the second derivative of $V(x)$ attains a positive value:

$$V''(x) = 6ax^{-4} - 2bx^{-3} \quad \xRightarrow{x=2a/b} \quad V''\left(\frac{2a}{b}\right) = 6a\frac{b^4}{(2a)^4} - 2b\frac{b^3}{(2a)^3} = \frac{3}{8}\frac{b^4}{a^3} - \frac{1}{4}\frac{b^4}{a^3} = \frac{1}{8}\frac{b^4}{a^3} > 0.$$

The Second Derivative Test implies that V has a local minimum at $x = 2a/b$ with the minimum value

$$V\left(\frac{2a}{b}\right) = a\frac{b^2}{(2a)^2} - b\frac{b}{2a} = \frac{1}{4}\frac{b^2}{a} - \frac{1}{2}\frac{b^2}{a} = -\frac{1}{4}\frac{b^2}{a}.$$

Let us call this value $V_{\min}$. Note that $V_{\min}$ is negative. Moreover, the potential has the limiting behaviour:

$$\lim_{x\to 0^+} [xV(x)] = a \lim_{x\to 0^+} \frac{1}{x} - b = \infty \quad \Rightarrow \quad \lim_{x\to 0^+} V(x) = \infty > V_{\min},$$

and

$$\lim_{x\to\infty} V(x) = a \lim_{x\to\infty} \frac{1}{x^2} - b \lim_{x\to\infty} \frac{1}{x} = a \cdot 0 - b \cdot 0 = 0 > V_{\min}.$$

Both being greater than $V_{\min}$, we can conclude that $V_{\min}$ is the absolute minimum of the potential.

The quadratic Taylor polynomial of V centred around $x = 2a/b$ is

$$\begin{aligned} T_2(x) &= V\left(\frac{2a}{b}\right) + V'\left(\frac{2a}{b}\right)\left(x - \frac{2a}{b}\right) + \frac{1}{2}V''\left(\frac{2a}{b}\right)\left(x - \frac{2a}{b}\right)^2 \\ &= -\frac{1}{4}\frac{b^2}{a} + 0\left(x - \frac{2a}{b}\right) + \frac{1}{2}\frac{1}{8}\frac{b^4}{a^3}\left(x - \frac{2a}{b}\right)^2 \\ &= \frac{b^4}{16a^3}x^2 - \frac{b^3}{4a^2}x. \end{aligned}$$

3) The *centroid* (point of balance) of a line segment is at its midpoint. The centroid of multiple line segments is obtained by taking the length-weighted average of centroids. Derive an integral formula for the coordinates $(\bar{x}, \bar{y})$ of the centroid of a curve $y = f(x)$, $a \le x \le b$ and use it to compute the centroid of the upper-semicircle of radius R centred at the origin.
[You may assume that $f(x)$ has a continuous derivative over $[a, b]$.]

Solution. Take an arbitrary positive integer n and divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal length using the points $x_0 := a$, $x_i := a + i\Delta x$, where $\Delta x := (b - a)/n > 0$ and $i = 1, \ldots, n$. Thus we have

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Introduce the notation $y_0 := f(a)$, $y_i := f(x_i)$ and $\Delta x_i := x_i - x_{i-1}$, $\Delta y_i := y_i - y_{i-1}$ for all $i = 1, \ldots, n$. Next, we create a broken line approximation to the curve $y = f(x)$ by connecting the points $(x_0, y_0) \to (x_1, y_1) \to \cdots \to (x_n, y_n)$. Let ℓ_i denote the length of the i-th line segment. The Pythagorean Theorem implies that

$$\ell_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}.$$

Clearly, we have $\Delta x_i = \Delta x$ and $\Delta y_i = f(x_i) - f(x_{i-1})$ so we can factor out Δx and write

$$\ell_i = \sqrt{1 + \left[\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right]^2}\,\Delta x.$$

The Mean Value Theorem guarantees the existence of a point $x_i^* \in (x_{i-1}, x_i)$ s.t. $f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$, hence

$$\ell_i = \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x.$$

The centroid of the i-th line segment is located at its midpoint, i.e.

$$(\bar{x}_i, \bar{y}_i) = \left(\frac{x_{i-1} + x_i}{2}, \frac{y_{i-1} + y_i}{2}\right)$$

To find the centroid of the broken line approximation we need to take the average of centroids of line segments weighted by their length. These weighted averages of the x- and y-coordinates of midpoints are

$$\bar{x}^{(n)} := \frac{\sum_{i=1}^n \bar{x}_i \ell_i}{\sum_{i=1}^n \ell_i} \quad \text{and} \quad \bar{y}^{(n)} := \frac{\sum_{i=1}^n \bar{y}_i \ell_i}{\sum_{i=1}^n \ell_i},$$

respectively. Note that the three sums

$$\sum_{i=1}^n \ell_i, \quad \sum_{i=1}^n \bar{x}_i \ell_i \quad , \sum_{i=1}^n \bar{y}_i \ell_i$$

that appear in the above quotients are Riemann Sums for the functions

$$\sqrt{1 + [f'(x)]^2}, \quad x\sqrt{1 + [f'(x)]^2}, \quad f(x)\sqrt{1 + [f'(x)]^2}.$$

Therefore as $n \to \infty$ we obtain the following integral formulas for the coordinates $(\bar{x}, \bar{y})$ of the centroid

$$\bar{x} = \frac{1}{L}\int_a^b x\sqrt{1+[f'(x)]^2}\,dx, \quad \bar{y} = \frac{1}{L}\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$$

where

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

is the arc length of the curve.

The upper-semicircle of radius R with centre at the origin is given by the graph of the function

$$f(x) = \sqrt{R^2 - x^2}, \quad -R \le x \le R.$$

We have

$$f'(x) = \frac{-x}{\sqrt{R^2-x^2}} \quad \Rightarrow \quad \sqrt{1+[f'(x)]^2} = \sqrt{1+\frac{x^2}{R^2-x^2}} = \frac{R}{\sqrt{R^2-x^2}}.$$

The arc length of the semicircle can be computed via the trigonometric substitution $x = R\sin\theta$ (giving $-\pi/2 \le \theta \le \pi/2$ and $dx = R\cos\theta\,d\theta$):

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_{-R}^{R} \frac{R}{\sqrt{R^2-x^2}}\,dx = \int_{-\pi/2}^{\pi/2} \frac{R}{R\cos\theta}\,R\cos\theta\,d\theta = R\int_{-\pi/2}^{\pi/2} d\theta = \pi R.$$

As for the remaining integrals, we have

$$\int_a^b x\sqrt{1+[f'(x)]^2}\,dx = \int_{-R}^{R} x\frac{R}{\sqrt{R^2-x^2}}\,dx = \left[-R\sqrt{R^2-x^2}\right]_{-R}^{R} = 0$$

and

$$\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx = \int_{-R}^{R} \sqrt{R^2-x^2}\frac{R}{\sqrt{R^2-x^2}}\,dx = R\int_{-R}^{R} dx = 2R^2.$$

Thus the centroid of the semicircle is located at

$$(\bar{x}, \bar{y}) = \left(\frac{0}{\pi R}, \frac{2R^2}{\pi R}\right) = \left(0, \frac{2R}{\pi}\right).$$

In other words, the centroid of the semicircle is $(2/\pi) \approx 63.66\%$ of the way up the radius along the y-axis.

4) Evaluate the definite integral $\displaystyle\int_0^{2\sqrt{3}} \frac{x+2}{\sqrt{4+x^2}}\,dx.$

Solution. There are several ways to evaluate this integral.

'Method' 1: Looking it up in tables of integrals. We can write

$$\int_0^{2\sqrt{3}} \frac{x+2}{\sqrt{4+x^2}}\,dx = \int_0^{2\sqrt{3}} \frac{x}{\sqrt{4+x^2}}\,dx + \int_0^{2\sqrt{3}} \frac{2}{\sqrt{4+x^2}}\,dx = [\sqrt{4+x^2}]_0^{2\sqrt{3}} + \int_0^{2\sqrt{3}} \frac{2}{\sqrt{4+x^2}}\,dx$$

The first term evaluates to $\sqrt{16} - \sqrt{4} = 2$ whereas the second term is a special case of the following formula [cf. Formula 25 on Reference Page 6 at the back of the Textbook]:

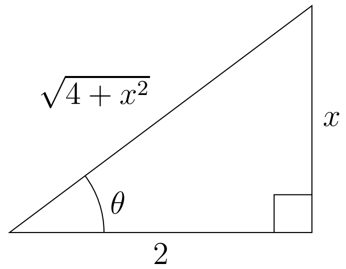
$$\int \frac{du}{\sqrt{a^2+u^2}} = \ln(u + \sqrt{a^2+u^2}) + C.$$

Thus we have

$$\int_0^{2\sqrt{3}} \frac{2}{\sqrt{4+x^2}}\,dx = 2\left[\ln(x+\sqrt{4+x^2})\right]_0^{2\sqrt{3}} = 2[\ln(2\sqrt{3}+4) - \ln 2] = 2\ln(\sqrt{3}+2)$$

and the integral in question evaluates to $2 + \ln(\sqrt{3}+2)$.

Method 2: Trigonometric Substitution. The factor $\sqrt{4+x^2}$ in the integrand suggests that a trigonometric substitution should be used. Such a square root appears as the length of hypotenuse in a right triangle with the legs having lengths of x and 2.



If θ denotes the angle opposite the leg of length x, then we can express x using θ by simply using the (right triangle) definition of the tangent function:

$$\tan\theta = \frac{x}{2} \quad \Rightarrow \quad \boxed{x = 2\tan\theta} \quad \Rightarrow \quad dx = \frac{2}{\cos^2\theta}\,d\theta.$$

Of course, we can also express the length of the hypotenuse in terms of θ as $\sqrt{4+x^2} = 2/\cos\theta$. Finally, we look at the limits of integration and see that $x = 0$ implies $\theta = 0$ and $x = 2\sqrt{3}$ implies $\theta = \pi/3$. Thus the substitution $x = 2\tan\theta$ results in the following

$$\begin{aligned}\int_0^{2\sqrt{3}} \frac{x+2}{\sqrt{4+x^2}}\,dx &= \int_0^{\pi/3} \frac{2\tan\theta+2}{\frac{2}{\cos\theta}}\frac{2}{\cos^2\theta}\,d\theta = \int_0^{\pi/3}\frac{2\sin\theta+2\cos\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/3}\frac{2\sin\theta}{\cos^2\theta}\,d\theta + \int_0^{\pi/3}\frac{2}{\cos\theta}\,d\theta \\ &= \left[\frac{2}{\cos\theta}\right]_0^{\pi/3} + \int_0^{\pi/3}\frac{2}{\cos\theta}\,d\theta = 2 + \int_0^{\pi/3}\frac{2}{\cos\theta}\,d\theta\end{aligned}$$

As for the remaining integral, we may use the tangent half-angle substitution $u = \tan(\theta/2)$ followed by partial fractions decomposition to get

$$\begin{aligned}\int_0^{\pi/3}\frac{2}{\cos\theta}\,d\theta &= \int_0^{1/\sqrt{3}} 2\frac{1+u^2}{1-u^2}\frac{2}{1+u^2}\,du = \int_0^{1/\sqrt{3}}\frac{4}{1-u^2}\,du = \int_0^{1/\sqrt{3}}\left(\frac{2}{1-u}+\frac{2}{1+u}\right)du \\ &= \left[-2\ln(1-u)+2\ln(1+u)\right]_0^{1/\sqrt{3}} = \left[2\ln\left(\frac{1+u}{1-u}\right)\right]_0^{1/\sqrt{3}} = 2\ln\left(\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}\right) = 2\ln(2+\sqrt{3})\end{aligned}$$

Therefore the integral in question evaluates to $2 + 2\ln(2+\sqrt{3})$.

Method 3: Hyperbolic Substitution. Inspired by the hyperbolic identity $\cosh^2 t - \sinh^2 t = 1$, we may substitute $x = 2\sinh t$ to get

$$\begin{aligned}\int_0^{2\sqrt{3}}\frac{x+2}{\sqrt{4+x^2}}\,dx &= \int_0^{\operatorname{arsinh}\sqrt{3}}\frac{2\sinh t+2}{2\cosh t}2\cosh t\,dt = \int_0^{\operatorname{arsinh}\sqrt{3}}(2\sinh t+2)\,dt \\ &= \left[2\cosh t + 2t\right]_0^{\operatorname{arsinh}\sqrt{3}} = 2\cosh\operatorname{arsinh}\sqrt{3} + 2\operatorname{arsinh}\sqrt{3} - 2\cosh 0 \\ &= 2\sqrt{1+\sinh^2\operatorname{arsinh}\sqrt{3}} + 2\operatorname{arsinh}\sqrt{3} - 2 = 2 + 2\operatorname{arsinh}\sqrt{3}.\end{aligned}$$

Note that the inverse hyperbolic sine function is $\operatorname{arsinh} x = \ln(x+\sqrt{x^2+1})$ for all x. At $x = \sqrt{3}$ we have $\operatorname{arsinh}\sqrt{3} = \ln(\sqrt{3}+2)$ confirming that the answers we got via all solution methods match.

5) To account for the seasonal variation, we may let the relative growth rate vary in the logistic differential equation. For example, consider the equation

$$P'(t) = k(t)P(t)\left(1-\frac{P(t)}{M}\right) \quad \text{with} \quad k(t) = 1+\cos t, \quad P(0) = P_0 \in (0, M).$$

Solve the above initial value problem for $P(t)$. What is $\lim_{t\to\infty} P(t)$?

Solution. The differential equation

$$P'(t) = k(t)P(t)\left(1 - \frac{P(t)}{M}\right)$$

is separable and so we can solve it by the general method (as seen in class and in Section 9.3 of the Textbook). Dividing both sides by $P(1 - P/M)$ followed by integration gives us

$$\int \frac{dP}{P(1 - P/M)} = \int k(t)dt$$

To evaluate the integral on the left-hand side, we use partial fractions and get

$$\frac{1}{P(1 - P/M)} = \frac{M}{P(M - P)} = \frac{1}{P} + \frac{1}{M - P}.$$

This enables us to rewrite the integral equation above as

$$\int \left(\frac{1}{P} + \frac{1}{M - P}\right) dP = \int k(t)dt$$

and thus

$$\ln|P| - \ln|M - P| = \int k(t)dt.$$

By using the properties of logarithms and absolute value, we obtain

$$\ln\left|\frac{M - P}{P}\right| = -\int k(t)dt,$$

which after exponentiation turns into the following

$$\left|\frac{M - P}{P}\right| = e^{-\int k(t)dt},$$

or equivalently,

$$\frac{M - P}{P} = \pm e^{-\int k(t)dt}.$$

Solving this equation for P, we find that

$$P(t) = \frac{M}{1 \pm e^{-\int k(t)dt}}.$$

In the current problem, we have $k(t) = 1 + \cos t$ and therefore

$$P(t) = \frac{M}{1 \pm e^{-\int (1+\cos t)dt}} = \frac{M}{1 \pm e^{-t-\sin t+C}}.$$

At $t = 0$, we must have $P(0) = P_0$ and therefore

$$P(0) = \frac{M}{1 \pm e^{C}} = P_0 \quad \Rightarrow \quad \pm e^{C} = \frac{M - P_0}{P_0}.$$

Thus the solution to the initial value problem is

$$P(t) = \frac{M}{1 + \frac{M-P_0}{P_0} e^{-t-\sin t}}$$

Since $0 < P_0 < M$ and $e^{-t-\sin t} \to 0$ as $t \to \infty$, we have

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{M}{1 + \frac{M-P_0}{P_0} e^{-t-\sin t}} = \frac{M}{1 + \frac{M-P_0}{P_0} \cdot 0} = M.$$

6) Solve the following initial value problem:

$$y''(x) + 2y'(x) + 5y(x) = 5 + 8e^x, \quad y(0) = 3, \quad y'(0) = 0.$$

[Hint: Use the method of undetermined coefficients.]

Solution. Recall that the general solution of such linear non-homogeneous second-order differential equations can be written as

$$y(x) = y_c(x) + y_p(x),$$

where $y_c(x)$ is the general solution of the homogeneous equation and $y_p(x)$ is a particular solution of the non-homogeneous equation (to be looked for via the method of undetermined coefficients). Thus we start by solving the homogeneous equation

$$y'' + 2y' + 5y = 0.$$

The auxiliary equation reads

$$r^2 + 2r + 5 = 0.$$

Notice that the discriminant is negative, namely $b^2 - 4ac = 2^2 - 4 \cdot 5 = -16 < 0$, therefore the quadratic formula yields two complex solutions

$$r_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 5}}{2} = \frac{-2 \pm 4i}{2} \quad \Rightarrow \quad r_1 = -1 + 2i, \; r_2 = -1 - 2i.$$

As shown in the lectures this means that the general solution of the homogeneous equation is

$$y_c(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x).$$

Now we turn to finding a particular solution $y_p(x)$. The right-hand side of the differential equation is a constant plus an exponential function therefore we may use the following test function

$$y_p(x) = A + Be^x.$$

Note that $y_p'(x) = Be^x$ and $y_p''(x) = Be^x$, hence substituting the test function into the non-homogeneous equation yields

$$(Be^x) + 2(Be^x) + 5(A + Be^x) = 5 + 8e^x.$$

Grouping the terms on the left-hand side leads to

$$5A + 8Be^x = 5 + 8e^x.$$

This holds for all x if and only if $A = B = 1$. Thus we have found that

$$y_p(x) = 1 + e^x$$

is a particular solution of the non-homogeneous equation $y'' + 2y' + 5y = 5 + 8e^x$. Adding to this the general solution y_c of the homogeneous equation produces the general solution of the non-homogeneous equation

$$y(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + 1 + e^x.$$

To determine the values of the constants c_1 and c_2 we need to compute the derivative of $y(x)$. We find that

$$y'(x) = e^{-x}([2c_2 - c_1] \cos 2x - [2c_1 + c_2] \sin 2x) + e^x.$$

Hence the initial values $y(0) = 3$, $y'(0) = 0$ impose the following constraints on c_1, c_2:

$$
\begin{aligned}
3 &= y(0) = c_1 + 2, \\
0 &= y'(0) = 2c_2 - c_1 + 1.
\end{aligned}
$$

The first equation implies that $c_1 = 1$ which, when plugged into the second equation, yields $0 = 2c_2 - 1 + 1$, thus $c_2 = 0$. Therefore the solution of the initial value problem $y'' + 2y' + 5y = 5 + 8e^x$, $y(0) = 3$, $y'(0) = 0$ is

$$y(x) = 1 + e^x + e^{-x} \cos 2x.$$